MATH5360 Game Theory
Exercise 4

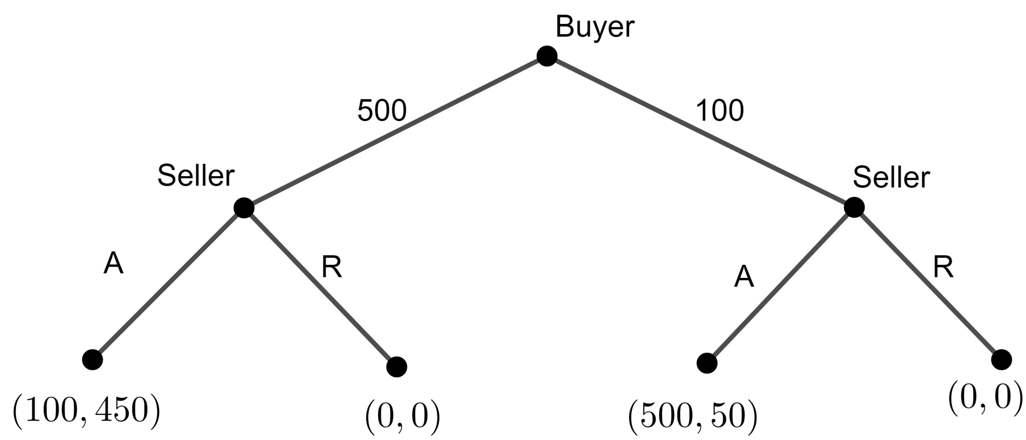
Assignment 4: 2, 4, 6, 8, 9, 10, 13 (Due: 6 April 2020 (Monday))

1. In a bargaining game, the buyer moves first by offering either \$500 or \$100 for a product that she values \$600. The seller, for whom the value of the object is \$50, responds by either accepting (A) or rejecting (R) the offer.

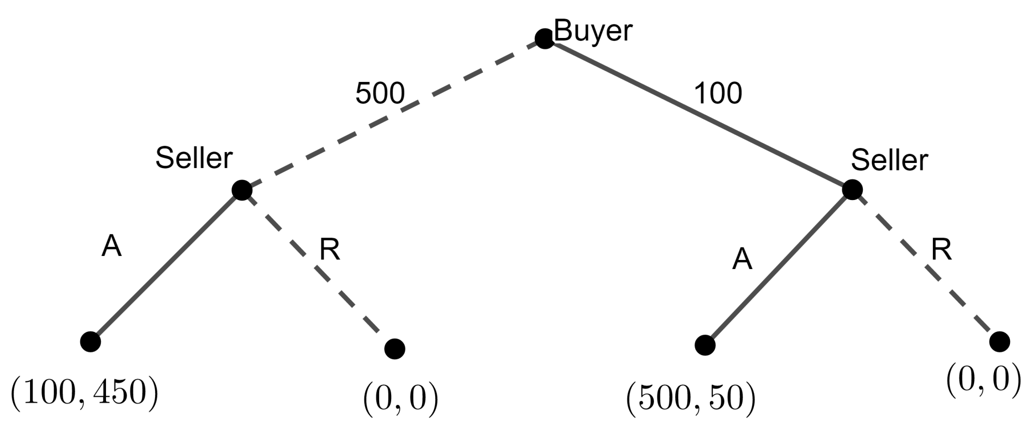
   (a) Draw the game tree of the bargaining game.

   (b) Use backward induction to solve the game.

   Answer: (a)



(b)

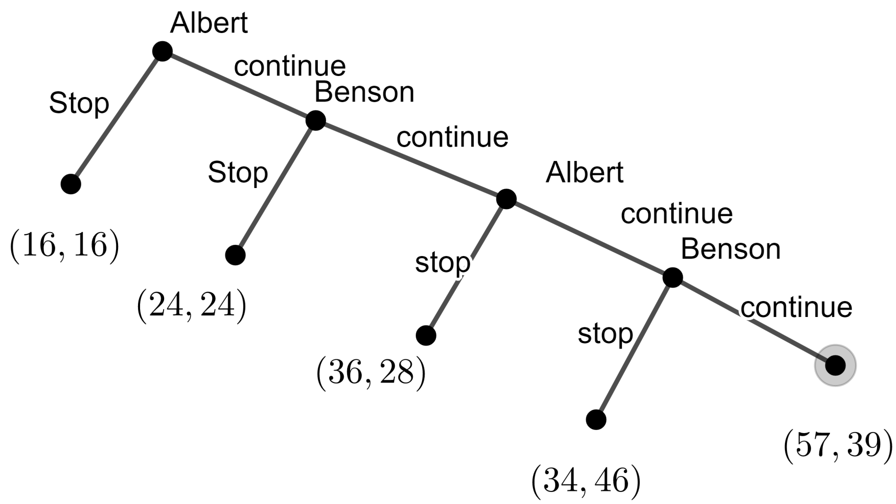


2. Albert and Benson start with \$16 in each of their piles. They take turns choosing one of two actions, continue or stop with Albert chooses first. Each time a player says continue, half of the amount in his pile will move to the other player's pile, and then extra \$16 will be added to his pile. The game automatically stop when the total amount in their piles reaches \$96.

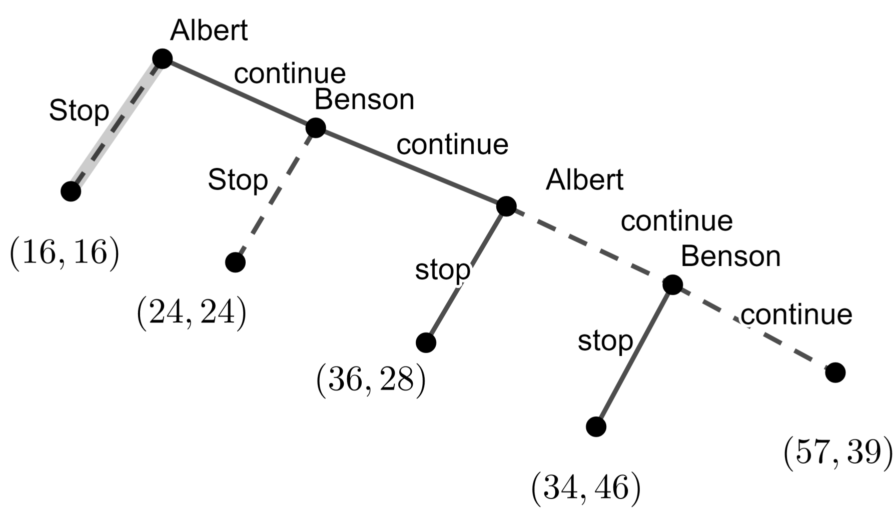
   (a) Draw the game tree of the game.

   (b) Use backward induction to solve the game and write down the payoffs of the players in the solution.

   Answer: (a)



(b)



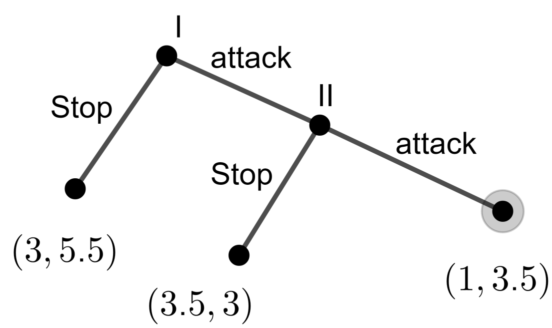
The payoff of Albert is 36 and the payoff of Benson is 28.

3. Armies I and II are fighting over an island initially held by a battalion of army II. Army I has 3 battalions and army II has 4, including the battalion occupying the island. Whenever the island is occupied by one army the opposing army can launch an attack with all its battalions. The outcome of the attack is that the army with more battalions will win and occupy the island with the surviving battalions which is equal to the difference of the number of battalions while the battalions of the loser will all be destroyed. The commander of each army is interested in maximizing the number of surviving battalions but also regards the occupation of the island as worth one and a half battalions.

   (a) Draw the game tree of the game.

   (b) Solve the game.

   Answer: (a)



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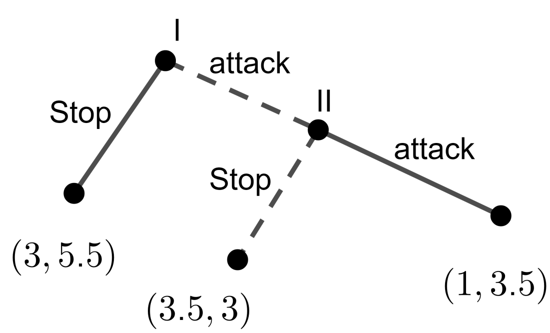


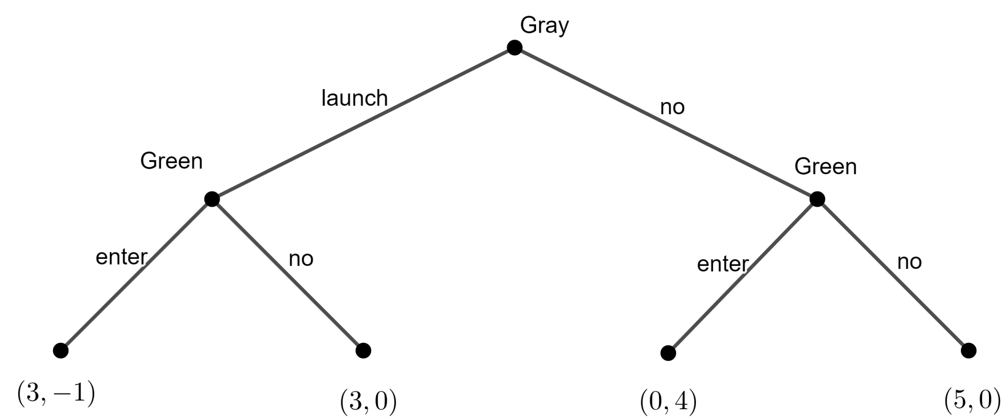
4. In a senate race game, a senate seat is currently occupied by Gray (the incumbent). A potential challenger for Gray's seat is Green. Gray moves first and decide whether to launch a preemptive advertising campaign and Green has to decide whether to enter the race. Green will win the senate seat only if Gray does not advertise and Green enters the race. Otherwise Gray will win the Senate Seat. Both Gray and Green value the senate seat as 5 units. However, 2 units of advertising cost will be deducted from the payoff of Gray if he launches the advertising and 1 unit of running cost will be deducted from the payoff of Green if he enters the race.

(a) Draw the game tree of the senate race game.

(b) Use backward induction to solve the game.

(c) Suppose Green does not know whether Gray has launched an advertising before he decides whether to enter the race. Draw the game tree and write down the strategic form of the game.

Answer: (a)



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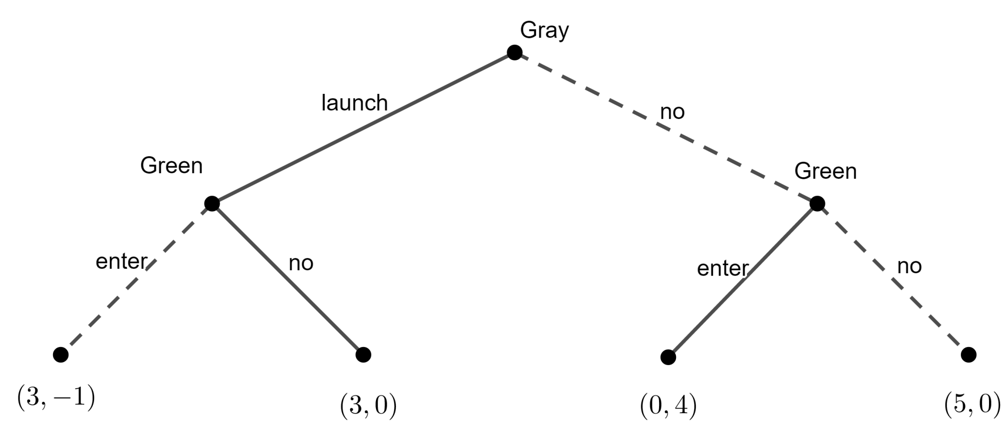
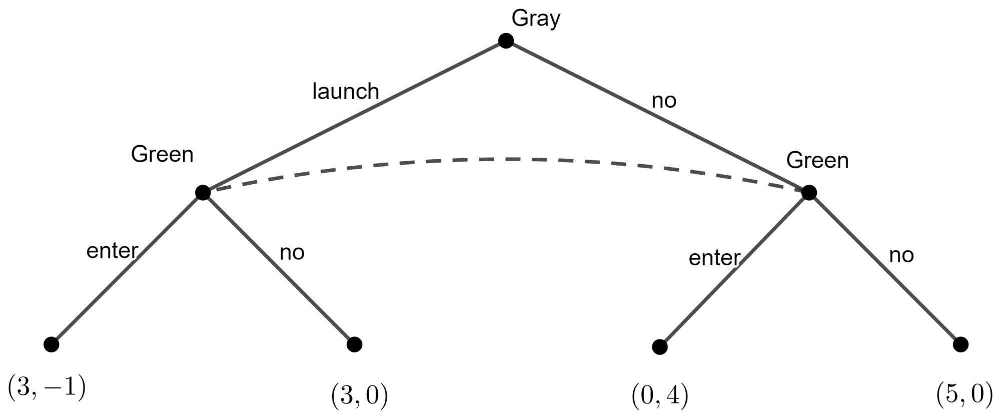


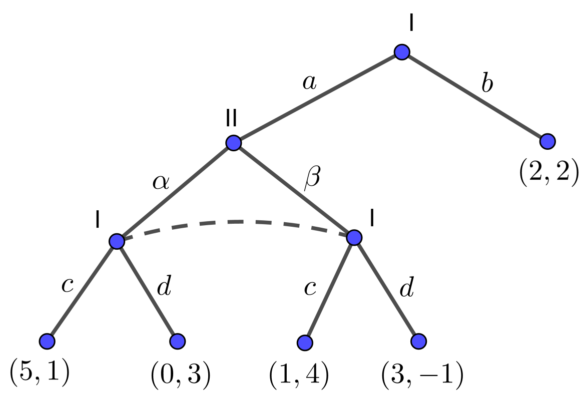
(c)



The game matrix is

$$\begin{array}{lc} Gray\backslash Green & \begin{array}{cc} enter & no \end{array} \\ launch & \multirow{2}{*}{$\begin{pmatrix} (3,-1) & (3,0) \\ (0,4) & (5,0) \end{pmatrix}$} \\ no & \end{array}$$

5. Consider the game tree



(a) Write down all pure strategies for Player I and Player II.

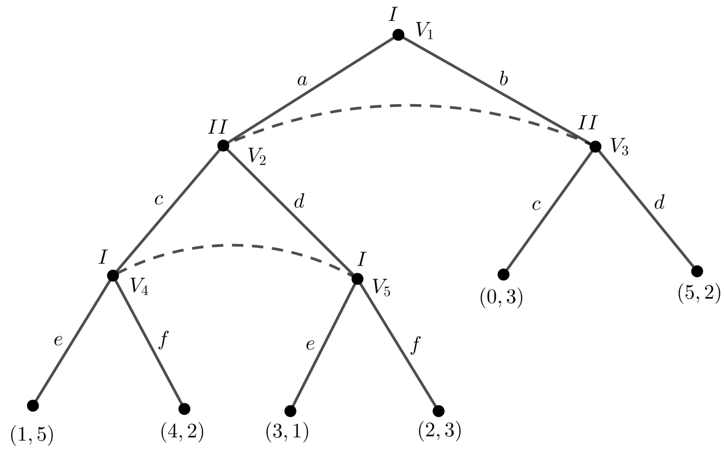
(b) Write down the strategic form (game bimatrix) of the game.

Answer: (a)I:ac,ad,b; II:α,β.

(b) The game matrix is

$$\begin{array}{l|cc} I\backslash II & \alpha & \beta \\ \hline ac & (5,1) & (1,4) \\ ad & (0,3) & (3,-1) \\ b & (2,2) & (2,2) \end{array}$$

6. Consider the game tree



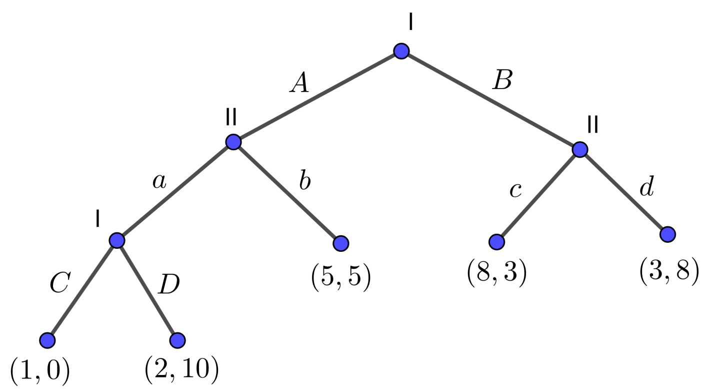
(a) Write down all pure strategies for Player I and Player II.

(b) Write down the strategic form (game bimatrix) of the game.

(c) Solve the subgame after Player I chooses a.

(d) Find all Nash equilibria of the game.

7. Consider the game tree



(a) Write down all pure strategies for Player I and Player II.

(b) Write down the strategic form (game bimatrix) of the game.

(c) Find all pure Nash equilibrium of the game.

(d) State whether each of the Nash equilibria is subgame perfect.

8. In a game show, there is \$5 in a green envelope and \$7 in a yellow envelope. A player Alan chooses an envelope and the amount inside the envelope is increased by \$4. Another player Bonnie, not knowing which envelope Alan has chosen, chooses an envelope and the amount inside the envelope is doubled. Then Alan, not knowing which envelope Bonnie has chosen, chooses envelope and gets the money inside. Bonnie will get the money inside the other envelope.

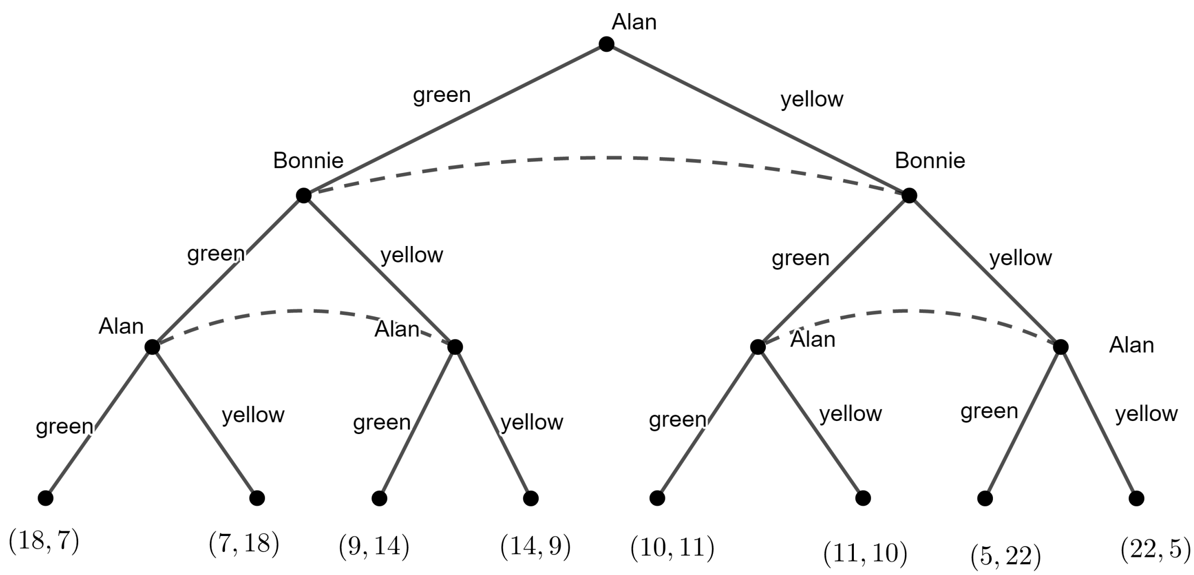
(a) Draw the game tree of the game.

(b) Write down all strategies of Alan and Bonnie.

(c) Write down the strategic form (game bimatrix) of the game.

(d) Find the Nash equilibrium of the game.

Answer: (a)



(b)Alan: GG,GY,YG,YY; Bonnie: G,Y.

(c) The game matrix is

$$\begin{array}{ll} Alan\backslash Bonnie & \begin{array}{cc} \quad G \quad & \quad Y \quad \end{array} \\ \begin{array}{l} GG \\ GY \\ YG \\ YY \end{array} & \begin{pmatrix} (18,7) & (9,14) \\ (7,18) & (14,9) \\ (10,11) & (5,22) \\ (11,10) & (22,5) \end{pmatrix} \end{array}$$

$$A = \begin{pmatrix} 18 & 9 \\ 7 & 14 \\ 10 & 5 \\ 11 & 22 \end{pmatrix},$$

$$B = \begin{pmatrix} 7 & 14 \\ 18 & 9 \\ 11 & 22 \\ 10 & 5 \end{pmatrix}.$$

Since the 2nd and 3rd rows of A is dominated by the last row of A, $\mathbf{p} = (x, 0, 0, 1-x)$. A and B can be reduced to

$$A' = \begin{pmatrix} 18 & 9 \\ 11 & 12 \end{pmatrix},$$

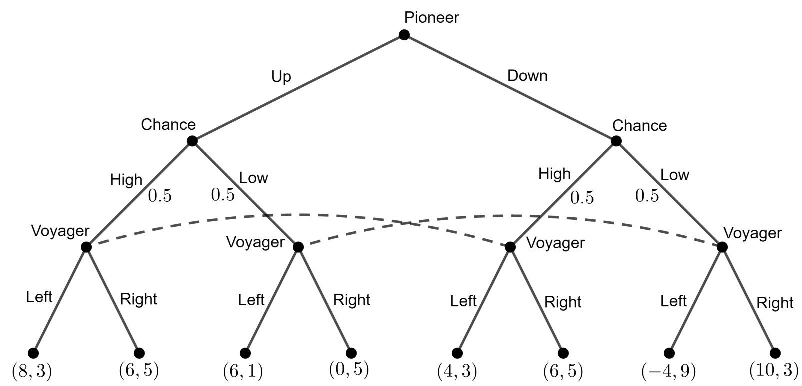
$$B' = \begin{pmatrix} 7 & 14 \\ 10 & 5 \end{pmatrix}.$$

$P = \{(x, y) : (x = 1 \cap 0.3 \le y \le 1) \cup (0 \le x \le 1 \cap y = 0.3) \cup (x = 0 \cap 0.3 \ge y \ge 0)\}$.

$Q = \{(x, y) : (0 \le x \le 5/12 \cap y = 1) \cup (x = 5/12 \cap 0 \le y \le 1) \cup (1 \ge x \ge 5/12 \cap y = 0)\}$.

$P \cap Q = \{(5/12, 0.3)\}$. Therefore, the game has one Nash equilibriums $(\mathbf{p}, \mathbf{q}) = \{((5/12, 0, 0, 7/12), (0.3, 0.7))\}$ and the payoff pair is $\{(11.7, 8.75)\}$.

9. Consider the game tree



After Pioneer has made the first move, the chance of High and Low are equal.

(a) Write down the strategies of Pioneer and Voyager.

(b) Write down the strategic form (game bimatrix) of the game.

(c) Find the Nash equilibrium of the game.

Answer: (a) Pioneer:Up,Down; Voyager:LL,LR,RL,RR

(b) The game matrix is

$$\begin{array}{ll} Pioneer\backslash Voyager & \begin{array}{cccc} LL & LR & RL & RR \end{array} \\ \begin{array}{l} Up \\ Down \end{array} & \begin{pmatrix} (7,2) & (4,4) & (6,3) & (3,5) \\ (0,6) & (7,3) & (1,7) & (8,4) \end{pmatrix} \end{array}$$

(c)

$$A = \begin{pmatrix} 7 & 4 & 6 & 3 \\ 0 & 7 & 1 & 8 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 4 & 3 & 5 \\ 6 & 3 & 7 & 4 \end{pmatrix}.$$

Since the 1st and 2nd columns of B are dominated by the 3rd and last columns of B respectively, $\mathbf{q} = (0, 0, y, 1 - y)$. A and B can be reduced to

$$A' = \begin{pmatrix} 6 & 3 \\ 1 & 8 \end{pmatrix},$$

$$B' = \begin{pmatrix} 3 & 5 \\ 7 & 4 \end{pmatrix}.$$

$P = \{(x,y) : (x = 1 \cap 0.5 \le y \le 1) \cup (0 \le x \le 1 \cap y = 0.5) \cup (x = 0 \cap 0.5 \ge y \ge 0)\}$.

$Q = \{(x,y) : (0 \le x \le 0.6 \cap y = 1) \cup (x = 0.6 \cap 0 \le y \le 1) \cup (1 \ge x \ge 0.6 \cap y = 0)\}$.

$P \cap Q = \{(0.6, 0.5)\}$. Therefore, the game has one Nash equilibriums $(\mathbf{p}, \mathbf{q}) = \{\big((0.6, 0.4), (0, 0, 0.5, 0.5)\big)\}$ and the payoff pair is $\{(4.5, 4.6)\}$.

10. Both Steve and Tony put \$1 to the pot. Steve draws a card from a winning card and a losing card randomly. Steve sees his card but keeps it hidden from Tony. Steve then bets or Checks. If Steve bets, he puts \$6 more into the pot and Tony, not knowing what card Steve has, must fold or call. If Tony folds, he loses \$1 to Steve no matter what card Steve has. If Tony calls, Steve wins \$7 from Tony if Steve has the winning card and Steve loses \$7 to Tony if Steve has the losing card. If Steve checks, his card is inspected. Steve wins \$1 from Tony if he has the winning card, and otherwise he loses \$1 to Tony.

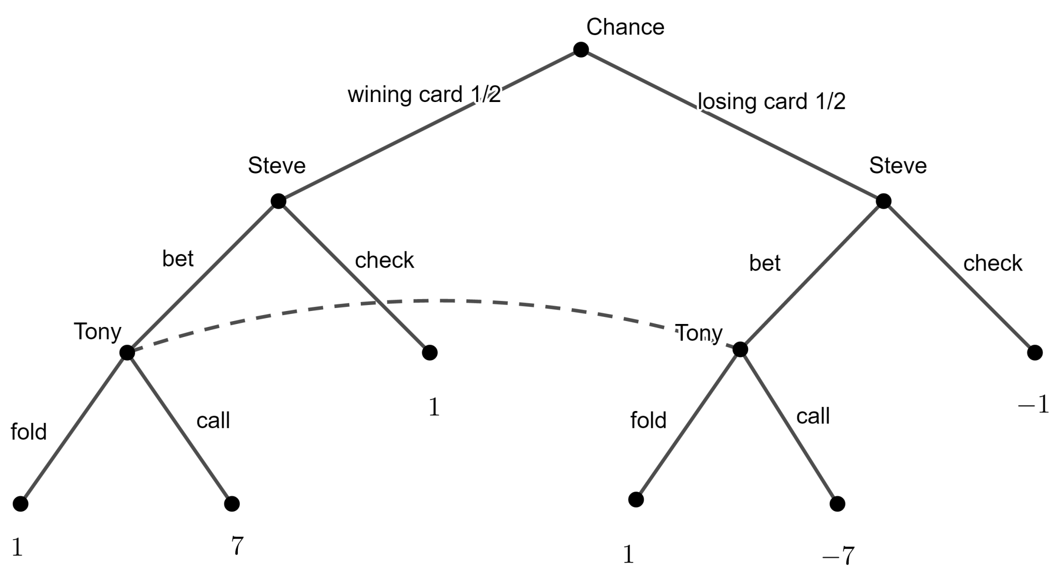
    (a) Draw the game tree of the game.

    (b) Write down the strategies of Steve and Tony.

    (c) Write down the strategic form (game matrix) of the game.

    (d) Solve the game.

    Answer: (a)



(b)Steve: BB,BC,CB,CC; Tony: fold,call.

(c) The game matrix is

$$\begin{array}{ll} Steve\backslash Tony & \begin{array}{cc} fold & call \end{array} \\ \begin{array}{l} BB \\ BC \\ CB \\ CC \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 1 & -3 \\ 0 & 0 \end{pmatrix} \end{array}$$

(d) The last two rows are dominated by the first row, thus the game matrix can be reduced to

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Therefore, the values of the game is $\nu = \frac{1\times 3-0}{1-0-0+3} = 3/4$ and the maximin strategy of Steve is $(3/4, 1/4, 0, 0)$, the minimax strategy of Tony is $(3/4, 1/4)$.

11. Players I and II play the following bluffing game. Each player bet \$1. Player I is given a card which is high or low; each is equally likely. Player I sees the card, player II doesn't. Player I can raise the bet to \$2 or fold. If player I folds, player I loses \$1 to player II. If player I raises, player II can call or fold. If player II folds, he loses \$1 to player I no matter what the card is. If player II calls, then player I wins \$2 from player II if his card is high and loses \$2 to player II if the card is low.

    (a) Draw the game tree of the game.

    (b) Write down all pure strategies of the players.

    (c) Write down the strategic form (game matrix) of the game.

    (d) Solve the game.

12. Two firms, an entrant (I) and an incumbent (II) play an market entry game. The entrant moves first, deciding to stay Out or to Enter the market. If the entrant stays Out, he gets a payoff of 0, while the incumbent gets the monopoly profit of 3. If the entrant Enters, the incumbent must choose between Fighting (so that both players obtain -1) or Accommodating (so that both players obtain the duopoly profit of 1).

    (a) Draw the game tree of the game.

    (b) Write down all pure strategies of the players.

    (c) Write down the strategic form of the game.

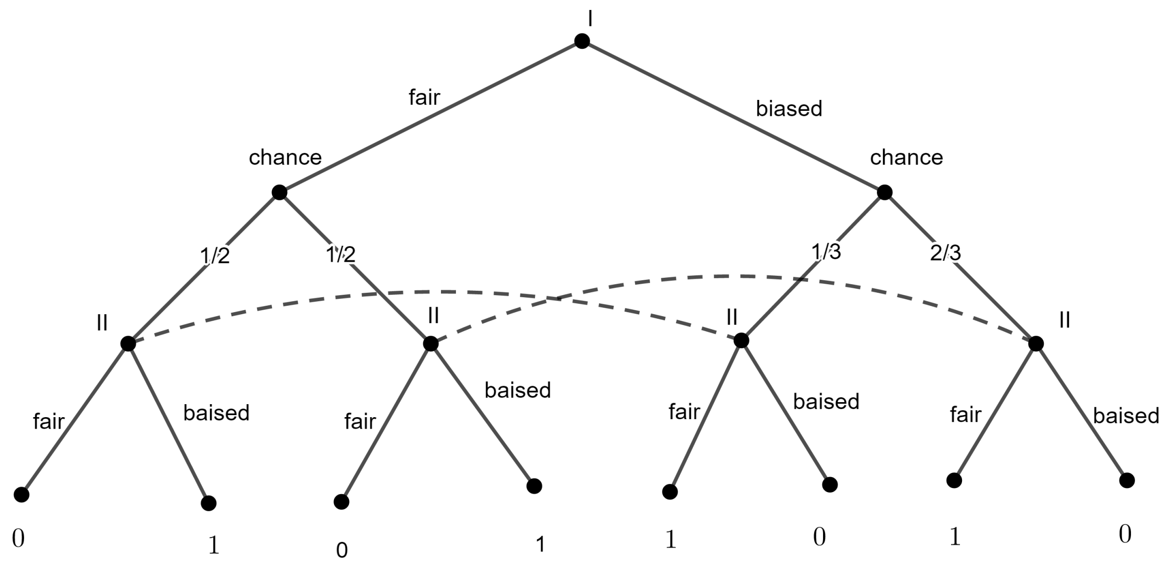
    (d) Solve the game.

13. Player I has two coins. One is fair (probability 1/2 of heads and 1/2 of tails) and the other is biased with probability 1/3 of heads and 2/3 of tails. Player I knows which coin is fair and which is biased. He selects one of the coins and tosses it. The outcome of the toss is announced to Player II. Then II must guess whether I chose the fair or biased coin. If II is correct there is no payoff. If II is incorrect, she loses 1 dollar.

    (a) Draw the game tree.

    (b) Solve the game.

    Answer: (a)



(b)The game matrix is

$$\begin{array}{c} I\backslash II \\ fair \\ biased \end{array} \begin{array}{c} \begin{matrix} FF & FB & BF & BB \end{matrix} \\ \begin{pmatrix} 0 & 1/2 & 1/2 & 1 \\ 1 & 1/3 & 2/3 & 0 \end{pmatrix} \end{array}$$

By deleting the dominated columns, we obtain

$$A' = \begin{pmatrix} 0 & 1/2 & 1 \\ 1 & 1/3 & 0 \end{pmatrix}.$$

Draw the graph of

$$\begin{cases} C_1 : v = 1 - x \\ C_2 : v = 1/6x + 1/3 \\ C_3 : v = x \end{cases}.$$

The lower envelope is shown in Figure 1. Solving

$$\begin{cases} C1 : v = 1 - x \\ C2 : v = 1/6x + 1/3 \end{cases},$$

we have $v = 3/7$ and $x = 4/7$. Hence the value of the game is $v(A) = 3/7$ and the optimal strategy for the row player is $(4/7, 3/7)$. Solving

$$\begin{pmatrix} 0 & 1/2 \\ 1 & 1/3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3/7 \\ 3/7 \end{pmatrix},$$

we have $y_1 = 1/7, y_2 = 6/7$. Therefore, the maximin strategy for player I is $(4/7, 3/7)$, the minimax strategy for player II is (1/7,6/7,0,0) and the value of the game is 3/7.

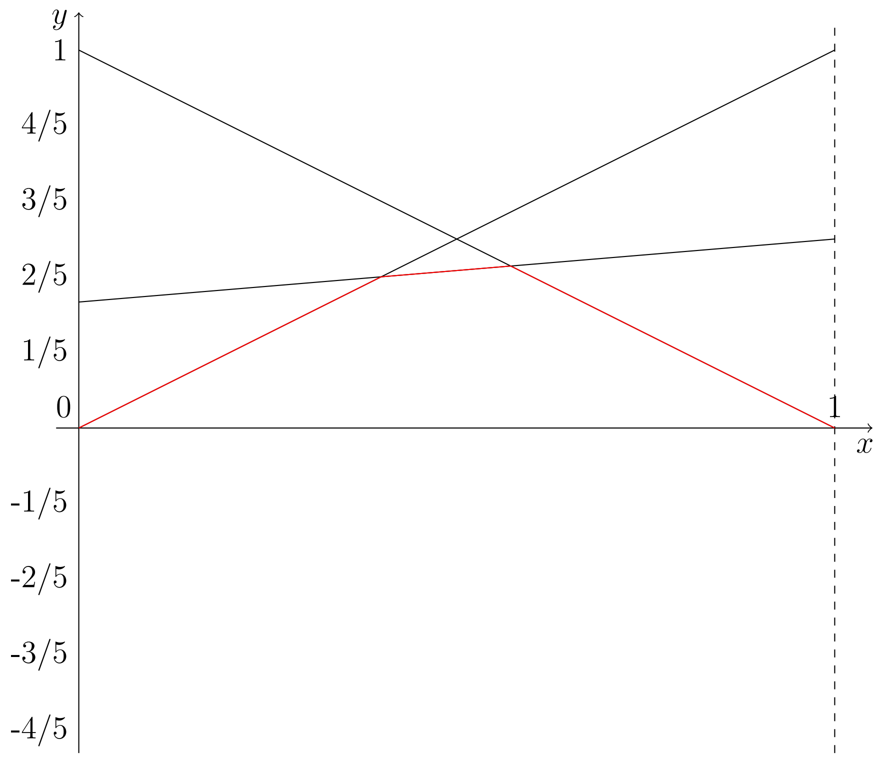


Figure 1:

14. A fair coin (probability 1/2 of heads and 1/2 of tails) is tossed and the outcome is shown to Player I. On the basis of the outcome of this toss, Player I decides whether to bet 1 or 2. Then Player II hearing the amount bet but not knowing the outcome of the toss, must guess whether the coin was heads or tails. Player II wins if her guess is correct and loses if her guess is incorrect. The absolute value of the amount won is the amount bet if the coin comes up tail and the amount bet plus 1 if the coin comes up heads.

    (a) Draw the game tree.

    (b) Write down the strategic form of the game.

    (c) Solve the game.

15. Coin A has probability 1/2 of heads and 1/2 of tails. Coin B has probability 1/3 of heads and 2/3 of tails. Player I must predict "heads" or "tails". If he predicts heads, coin A is tossed. If he predicts tails, coin B is tossed. Player II is informed as to whether I's prediction was right or wrong (but she is not informed of the prediction or the coin that was used), and then must guess whether coin A or coin B was used. If Player II guesses correctly she wins 1 dollar from Player I. If Player II guesses incorrectly and Player I's prediction was right, Player I wins 2 dollars from Player II. If both are wrong there is no payoff.

    (a) Draw the game tree of the game.

    (b) Write down the strategic form of the game.

    (c) Solve the game.

16. Consider the two games

$$G_1 = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } G_2 = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}.$$

One of these games is chosen to be played at random with probability 1/3 for G_1 and probability 2/3 for G_2. The game chosen is revealed to Player I but not to Player II. Then Player I selects a row, 1 or 2, and simultaneously Player II chooses a column, 1 or 2, with payoff determined by the selected game.

   (a) Draw the game tree.

   (b) Solve the game.

17. Player I draws a card at random from a full deck of 52 cards. After looking at the card, he bets either 1 or 5 that the card he drew is a face card (king, queen or jack). Then Player II either concedes or doubles. If she concedes, she pays Player I the amount bet (no matter what the card was). If she doubles, the card is shown to her, and Player I wins twice his bet if the card is a face card, and loses twice his bet otherwise.

   (a) Draw the game tree.

   (b) Write down the strategic form of the game.

   (c) Solve the game.

18. Player II must count from n down to zero by subtracting either one or two at each stage. Player I must guess at each stage whether Player II is going to subtract one or two. If Player I ever guesses incorrectly at any stage, the game is over and there is no payoff. Otherwise, if Player I guesses correctly at each stage, he wins 1 from Player II. Let G_n denote this game, and use the initial conditions $G_0 = (1)$ and $G_1 = (1)$. Let v_n be the value of G_n.

   (a) Find v_3, v_4 and v_5.

   (b) Find v_n. (You may use F_n to denote the Fibonacci sequence, $0, 1, 1, 2, 3, 5, 8, 13, \cdots$, with definition $F_0 = 0, F_1 = 1$, and for $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.)

19. There is one point to go in the match. The player that wins the last point while serving wins the match. The server has two strategies, high and low. The receiver has two strategies, near and far. The probability the server wins the point is given in the accompanying table.

	near	far
high	0.8	0.5
low	0.6	0.7

If the server misses the point, the roles of the players are interchanged and the win probabilities for given pure strategies are the same for the new server. Find optimal strategies for server and receiver, and find the probability the server wins the match.

20. Player I tosses a coin with probability p of heads. For each $k = 1, 2, \cdots$, if Player I tosses k heads in a row he may stop and challenge Player II to toss the same number of heads; then Player II tosses the coin and wins if and only if he tosses k heads in a row. If Player I tosses tails before challenging Player II, then the game is repeated with the roles of the players reversed. If neither player ever challenges, the game is a draw.

    (a) Solve the game when $p = 1/2$.

    (b) For arbitrary p, find the optimal strategies of the players and the probability that Player I wins.